Number of spanning clusters at the high-dimensional percolation thresholds

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A scaling theory is used to derive the dependence of the average number $\langle k \rangle$ of spanning clusters at threshold on the lattice size L. This number should become independent of L for dimensions $d < 6$ and vary as $\ln L$ at $d = 6$. The predictions for $d > 6$ depend on the boundary conditions, and the results there may vary between L^{d-6} and L^0. While simulations in six dimensions are consistent with this prediction [after including corrections of order $\ln(\ln L)$], in five dimensions the average number of spanning clusters still increases as $\ln L$ even up to $L = 201$. However, the histogram $P(k)$ of the spanning cluster multiplicity does scale as a function of $kX(L)$, with $X(L) = 1 + \text{const}/L$, indicating that for sufficiently large L the average $\langle k \rangle$ will approach a finite value: a fit of the five-dimensional multiplicity data with a constant plus a simple linear correction to scaling reproduces the data very well. Numerical simulations for $d > 6$ and for $d = 4$ are also presented.

I. INTRODUCTION AND THEORY

We are interested in site percolation on a finite hypercubic lattice in d dimensions, of linear size L. The number of spanning clusters when L is of the order of the percolation correlation length ξ away from the percolation threshold p_c was discussed originally in Ref. [1]. The purpose of this paper is to discuss the average number of spanning clusters, $\langle k \rangle$, at $p = p_c$, when ξ is infinite, and all the critical quantities behave as powers of L. As we discuss below, the number of spanning clusters has been the topic of much discussion in the literature. For example, although intuitively one would think that in two dimensions there exists only one spanning cluster, in fact (for appropriate boundary conditions) there is a whole distribution of the sizes of such clusters [2]. In addition, we show below that for $d < 6$ the average number of such clusters is finite, in direct relation with the validity of the hyperscaling relations among critical exponents. For $d > 6$ hyperscaling is violated, dangerous irrelevant variables must be introduced, and the result for $\langle k \rangle$ becomes ambiguous, depending on the boundary conditions. In fact, the theory for that case is yet incomplete, leaving an open challenge for future research.

We start with a theoretical discussion. The percolation "order parameter" P_{span} is usually defined as the probability that a site belongs to *any* spanning cluster (i.e., to the union of *all* spanning clusters). For a finite sample, P_{span} is related to the cluster size distribution function $n_s(p,L)$ (defined as the average number per site of clusters containing s sites) via the sum rule [3]

$$P_{span}(p,L) = p - \sum_s s n_s(p,L), \tag{1}$$

where the sum over s excludes all clusters which span the lattice. The exact details depend on the definition of "spanning"—e.g. along how many directions should the cluster connect opposite faces of the hypercube. However, these details do not matter for the scaling arguments presented below.

For a finite lattice, the sum in Eq. (1) goes up to $s_{max}(L)$, which is of the same order as the average mass of a single spanning cluster, $s(L)$. Since the total mass of *all* spanning clusters is given by $L^d P_{span}$, this implies that the average number $\langle k \rangle$ of spanning clusters is given by

$$\langle k \rangle \propto L^d P_{span}/s(L). \tag{2}$$

This relation should hold at all dimensions. The proportionality constant in Eq. (2) (which results, among other things, from dividing averages rather than averaging the ratio and on the detailed definition of "spanning" in a finite sample) may depend on the boundary conditions and on other details.

One way to derive $s(L)$ is to use the pair connectivity function, $G(r)$. This function yields the probability that a site at distance r from the origin is connected to the same cluster as the origin. Assuming that the site at the origin belongs to any spanning cluster, with probability P_{span}, the density profile of the cluster is given by $\rho(r) = G(r)/P_{span}$ [4,5]. Thus,

$$s(L) \propto \int_0^L d^d r \, G(r)/P_{span} = S(L)/P_{span}, \qquad (3)$$

where $S(L) = \int_0^L d^d r \, G(r)$ is proportional to the mean cluster size:

$$S \propto \sum_s s^2 n_s(p,L) / \sum_s s n_s. $$

(These two quantities are equal in the thermodynamic limit, $L \to \infty$.) Thus, we come to the fundamental relation

$$\langle k \rangle \propto L^d P_{span}^2 / S(L). \qquad (4)$$

A naive finite-size scaling theory would predict that at $p = p_c$ one has $P_{span} \propto L^{-\beta/\nu}$ and $S(L) \propto L^{\gamma/\nu}$ [3]. Thus, one concludes that

$$\langle k \rangle \propto L^{d-(2\beta+\gamma)/\nu}. \qquad (5)$$

For $d < 6$, hyperscaling implies that $d\nu = 2\beta + \gamma$, and therefore $\langle k \rangle$ is asymptotically independent of L. In fact, the combination of amplitudes which appears in Eq. (2) is universal, depending only on the type of boundary conditions [6]. As we discuss below, numerical estimates for $\langle k \rangle$ indeed approach a constant for $d \leq 4$. However, data for $d = 5$ require further discussion. Although not a major purpose of this paper, our discussion below should also serve as a warning and as a guideline for future numerical simulations in such high dimensions. In fact, some data remain ambiguous unless one uses available theoretical information or unless one performs simulations on much larger scales than presently possible.

At $d = 6$, many power laws are modified by logarithmic corrections. In fact [7],

$$P_{span} \propto L^{-2}(\ln L)^{11/21},$$

$$S(L) \propto L^2(\ln L)^{1/21}. \qquad (6)$$

Using our basic result, Eq. (4), we thus find

$$\langle k \rangle \propto \frac{L^6 P_{span}^2}{S(L)} \propto \ln L, \quad d = 6. \qquad (7)$$

Further analysis shows that the coefficient of proportionality here (and also the result for $\langle k \rangle$ for $d < 6$) is a universal number [6]. Corrections to the above result will involve $\ln(\ln L + \text{const})$. Figure 1 shows that the data in six dimensions, which were fitted by the square of $\ln L$ in Ref. [8], can also be fitted to a simple logarithm plus a $\ln \ln L$ correction to scaling, $\langle k \rangle = A \ln L + B \ln \ln L + \cdots$. The two data sets refer, respectively, to free boundary conditions (FBC's) and to mixed boundaries (MBC's)—i.e., helical in $d-2$ directions and free in the remaining two. In both cases we obtain a good fit for $L > 10$, so that the trend appears independent of boundary conditions, as one would expect. The coefficients of the fits are $A = 11.9$, $B = -17$ (MBC) and $A = 2.38$, $B = -4.33$ (FBC).

For $d > 6$, hyperscaling is broken, and one has the mean-field exponents $\beta = \gamma = 2\nu = 1$. A simple scaling in which ξ is replaced by L [1] would give $P_{span} \propto L^{-2}$, $S(L) \propto L^2$, $s(L) \propto L^4$, and
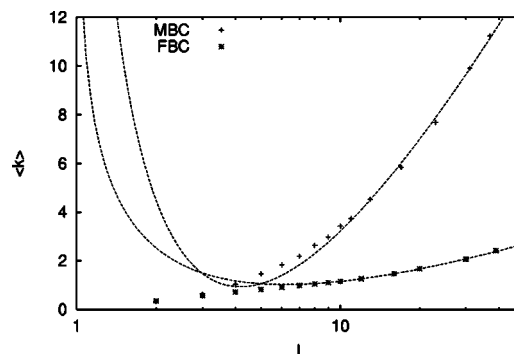

FIG. 1. Average number $\langle k \rangle$ of simultaneously spanning clusters at the six-dimensional percolation threshold, for FBC's ($*$) and MBC's ($+$) (see text). The lines are fits to $\langle k \rangle = A \ln(L) + B \ln(\ln L)$; we used the data of Ref. [8] plus additional simulations for small L with the number of samples variable from 1000 to 50 000.

$$\langle k \rangle \propto L^{d-6}. \qquad (8)$$

This simple result, which already appeared in Ref. [3], has also been proved by Aizenman [2] for the case of bulk boundary conditions, where a spanning cluster connects two opposite faces of the box of size L under the condition that sites in the box can also be connected by paths outside the box.

However, other forms of scaling may be possible, which may depend on the boundary conditions. We will discuss the different scaling approaches within the framework of the renormalization group (RG) theory (e.g., Ref. [7]). This theory is conveniently discussed using the "free energy" F, equal to the generating function of the cluster distribution function,

$$F(p,h,L) = \sum_s n_s(p,L) e^{-sh}, \qquad (9)$$

which is related to the free energy of the q-state Potts model for $q \to 1$. Then P_{span} and $S(L)$ are the first and second derivatives of the singular part of F with respect to h. After ℓ RG iterations, this singular part becomes

$$f(p,h,w,L) = e^{-d\ell} f(t(\ell),h(\ell),w(\ell),L/e^\ell), \qquad (10)$$

where $t = p - p_c$. Here, w represents the probability for having a threefold vertex at a site on a cluster. Although irrelevant in the RG sense, this variable must be included in the analysis, since the "free energy" may depend on it in a singular way, causing the breakdown of hyperscaling. This is why w is called "a dangerous irrelevant variable." For a finite sample close to p_c, $L \ll \xi$, iteration until $e^\ell = L$ yields the L dependence of the quantities discussed above.

For $d < 6$, $w(\ell)$ approaches a finite fixed-point value, and one ends up with hyperscaling and with the conclusion that k approaches a finite constant. For $d > 6$, one ends up with

$$f(t,h,w,L) = L^{-d} f(tL^2, hL^{d/2+1}, wL^{3-d/2}, 1). \qquad (11)$$

However, as stated above, now w turns out to be a "dangerous irrelevant variable." One way to see this is to consider the infinite sample. After eliminating the fluctuations, f is given by the minimum of the Landau free energy,
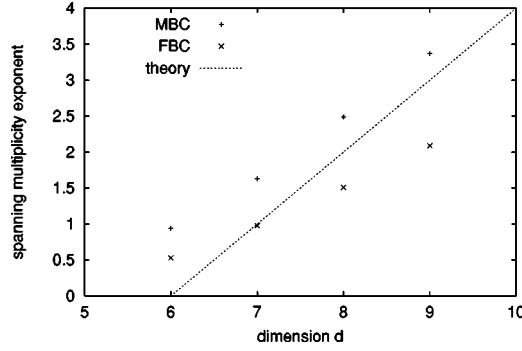
FIG. 2. Effective exponents for the scaling with L of the spanning cluster multiplicity (from Ref. [8]), as a function of the spatial dimension d. The dashed line represents the theoretical prediction from Eq. (8).

$$f = \min_{P}[tP^2 + wP^3 - hP], \tag{12}$$

while the order parameter P_{span} is equal to the value of P which minimizes this free energy. Replacing P by $Qw^{-1+x/3}$, with an arbitrary exponent x, it is easy to see that f obeys the scaling relation

$$f(t,h,w) = w^{-2+x}f(tw^{-x/3}, hw^{1-2x/3}, 1). \tag{13}$$

In particular, the minimization with respect to Q now yields the equation $3Q^2 + 2Qtw^{-x/3} - hw^{1-2x/3} = 0$, leading to the scaling form

$$f(t,h,w) = w^{-2}t^3g(hw/t^2), \tag{14}$$

which is completely independent of the arbitrary exponent x. This ambiguity stems from the fact that the Landau free energy is calculated for an infinite system.

Unlike the above result for the infinite system, the exponent x does persist for *finite* samples. In fact, the theoretical predictions for $\langle k \rangle$ depend crucially on x, and this remains an open challenge for future research. In this case, combining the scaling with L from Eq. (11) with Eq. (13), we end up with

$$f(t,h,w,L) = w^{x-2}L^{-6+3x-dx/2}$$
$$\times f(tw^{-x/3}L^{2-x+dx/6}, hw^{1-2x/3}L^{4-2x+dx/3}, 1, 1), \tag{15}$$

and the value of x must follow from the boundary condition, which breaks the scale invariance reflected in Eq. (14). Taking derivatives with respect to h and setting $t = 0$, this implies that

$$P_{span} \propto L^{-2+x-dx/6},$$

$$S \propto L^{2-x+dx/6},$$

$$s(L) \propto L^{4-2x+dx/3}. \tag{16}$$

The naive scaling result of Eq. (8) is obtained with the simple choice $x = 0$.

Figure 2 compares Eq. (8) with the numerically estimated effective exponents. Clearly, there exist some discrepancies
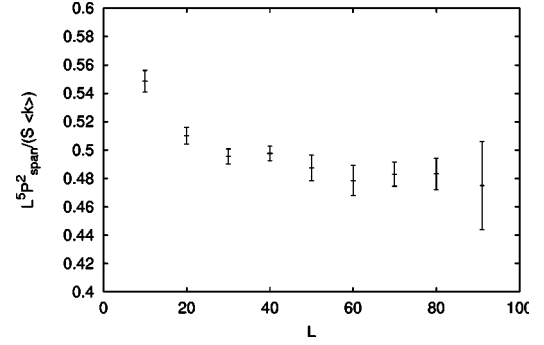

FIG. 3. Numerical verification of our relation, Eq. (4), for five-dimensional (5D) percolation with MBC's.

between theory and simulations. However, these discrepancies do not worry us much since the lattices for $d > 6$ are very small in their linear dimension L. This is confirmed by the fact that, for $d > 6$, the results for the exponents strongly depend on the type of boundary conditions one chooses. Also, our attempts to add corrections to scaling to the leading term of the fit ansatz did not improve the situation. At present, the numerical results cannot clearly confirm that $x = 0$, as would be required if Aizenman's [2] result also applies for our boundary conditions (note that his proof works under somewhat different conditions). Thus, the value of x remains to be determined in the future.

A second possibly relevant case is $x = 2$. In this case, f has the form

$$f(t,h,w,L) = L^{-d}\mathcal{F}(w^{-2/3}tL^{d/3}, hw^{-1/3}L^{2d/3}, 1, 1), \tag{17}$$

yielding $D = 2d/3$ and P_{span} and S given by

$$P_{span} = L^{-d/3}f_1(w^{-2/3}tL^{d/3}), \tag{18}$$

$$S = L^{d/3}f_2(w^{-2/3}tL^{d/3}). \tag{19}$$

In the limits $t \to 0$, $L \to \infty$, these equations behave as

$$P_{span} \propto S^{-1} \propto t \propto \xi^{-2}, \quad L = \infty, \tag{20}$$

$$P_{span} \propto S^{-1} \propto L^{-d/3}, \quad \langle k \rangle \propto \text{const}, \quad t = 0, \tag{21}$$

and

$$P_{span} \propto S^{-1} \propto L^{-y}, \quad \langle k \rangle \propto L^{d-3y},$$

$$tL^y = \text{const} < 0, \quad y \leqslant d/3. \tag{22}$$

(Remember that these equations only apply for $d > 6$.) The scaling behavior of Eqs. (20)–(22) for $y = 2$ is analogous to that found by Chen and Dohm [9] for the ϕ^4 Ising model with periodic boundary conditions and has recently been used to find an upper bound for the number of spanning clusters [8]. As seen from Fig. 2, our data for $t = 0$ do not seem to agree with the prediction that $\langle k \rangle$ approaches a finite constant. However, our samples are small, and the situation for $d > 6$ requires more studies. Very recent simulations show that P_{span} differs appreciably from the relative size of the largest cluster [10], but that is not enough to conclude that the spanning cluster multiplicity diverges for $d > 6$. Thus, the
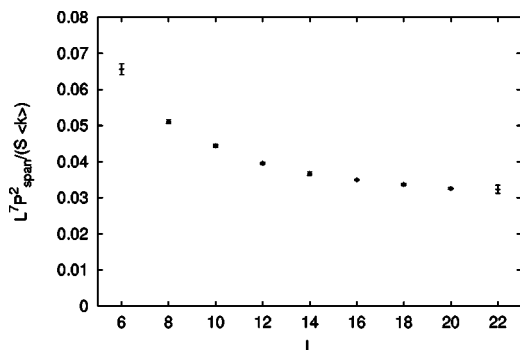
FIG. 4. As Fig. 3, but for 7D percolation with FBC's.

true behavior of $\langle k \rangle$ for $d > 6$ remains an open issue. We hope that the present paper will stimulate both numerical and theoretical discussions of this question.

II. RESULTS OF THE SIMULATIONS

We now review in detail our numerical simulations. The first numerical studies on this topic were performed by de Arcangelis [11].

An important issue concerns the validity of our Eq. (4), which is a general result that establishes a link between the order parameter P_{span}, the mean cluster size S, and the spanning cluster multiplicity $\langle k \rangle$. Figures 3 and 4 show numerical tests of this relation. We analyzed one case below the upper critical dimension $d_c = 6$—i.e., 5D with MBC's—and one case above d_c—i.e., 7D with FBC's. In both cases we have calculated the ratio $L^d P_{span}^2 / (S\langle k \rangle)$. From our equation (4) we expect that for L large this ratio converges to a constant, and both our figures confirm this expectation. We stress that the S that we calculate through our simulations differs from the standard definition of $S = \Sigma_s s^2 n_s(p, L) / \Sigma_s s n_s$ by the absence of the denominator. The latter is smaller than 1, as it is the density of occupied sites which belongs to finite clusters, so the real value of the ratio $L^d P_{span}^2 / (S\langle k \rangle)$ would be smaller
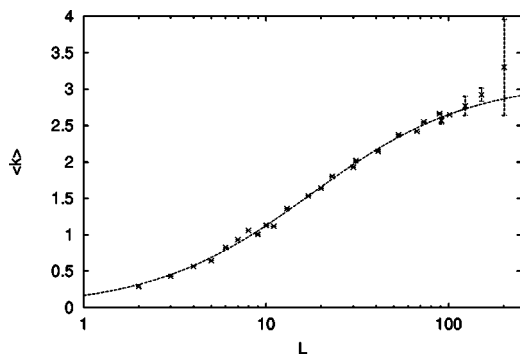

FIG. 5. Five-dimensional problem, MBC's. The rightmost error bar for $L = 201$ comes from 10 samples, the two smaller error bars shown from 100 samples. Most of the data use 1000 samples with error bars too small to be shown. There is no evidence for a plateau; nevertheless, the trend of the data is beautifully reproduced just by adding a simple nonlogarithmic scaling correction (dashed line in the plot).
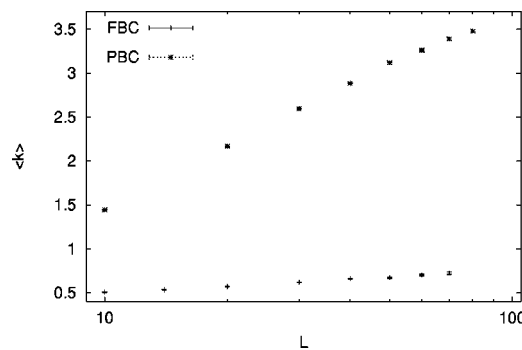

FIG. 6. Five-dimensional data of the spanning cluster multiplicity at threshold for FBC's (+) and PBC's (∗). One still finds an increase with L, although for PBC's it seems that the data curve starts bending at high L. The number of samples goes from 3000 to 50 000 for FBC's and from 10 000 to 50 000 for PBC's.

than the one we show in Figs. 3 and 4. On the other hand, the functional dependence on L of the ratio is the same in both cases.

Let us now concentrate on the trend of the multiplicity when the linear dimension L of the lattice varies. In less than five dimensions, Ref. [8] found an asymptotically constant number of spanning clusters for both theory and simulation. Thus the real problem is five dimensions, which we discuss now.

Figure 5 shows new data for L^5 sites, extending up to $L = 201$, the largest five-dimensional system known to us from direct simulations (176^5 was simulated in Ising models [12]). They are still compatible with the number of spanning clusters increasing as $\ln L$. These data use MBC's, but similar proportionalities to $\ln L$ were found also with free and periodic boundary conditions (PBC's, which means helical boundaries in $d-1$ directions), which are illustrated in Fig. 6. By looking at the data corresponding to periodic boundaries, however, it seems that the curve smoothly bends towards the end, which might indicate the beginning of a crossover.

For comparison, Fig. 7 shows four-dimensional simulations with MBC's. There we see for small $L \leq 12$ a logarithmic increase, followed by a crossover region extending over one decade in L and ending finally in the theoretically pre-
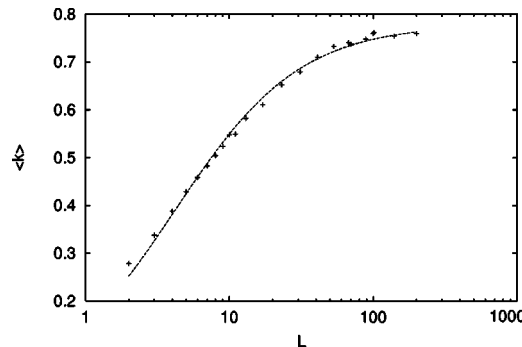

FIG. 7. Four-dimensional solution, MBC's. The spanning cluster multiplicity increases in a wide range of L, but attains finally a plateau for $L > 100$. The dashed curve is the fit with the simple correction to scaling ansatz $aL/(L+b)$. For each L we took mostly 1000 samples.

dicted plateau near $L=200$. Further simulations with PBC's (not shown) confirm this trend. These four-dimensional results are a nice example for the need of large lattices in simulations: only above $L=100$ is the theory confirmed. Comparing Figs. 5 and 7 we may hope that also in five dimensions a crossover would be seen towards a plateau, if we could simulate larger lattices than present world records [13].

There is a way, however, to clarify the issue. Instead of looking merely at the average number $\langle k \rangle$ of spanning clusters (as in all of the above discussion), we can analyze the histogram $P(k)$ of that number. Its tail for large k was already shown to be consistent with theory [14]: $-\ln P(k \to \infty) \propto k^{d/(d-1)}$ for $2 \leqslant d \leqslant 7$. If $\langle k \rangle$ were size independent, on a lattice of linear dimension L we would also expect $P(k,L) = P(k)$; i.e., the histogram is as well size independent. Finite-size scaling for large L would nevertheless allow correction factors like $X(L)=1+\text{const}/L$, so that the histogram is indeed a function $P(kX(L))$. On the other hand, $P(kX(L))$ is normalized to some L-independent number C (we chose here $C=1000$), and therefore

$$\sum_{k=0}^{\infty} P(kX(L)) = C. \tag{23}$$

If we approximate the sum with the integral over k, we can perform the change of variable $j=kX(L)$, so to get

$$\frac{1}{X(L)} \sum_{j=0}^{\infty} P(j) = C. \tag{24}$$

Equation (24) cannot be right as it stands, because the left-hand side is a function of L, while the right-hand side is not. That means that the histogram $P(k,L)$ is not a scaling function of the variable $j=kX(L)$, but that $P(k,L) = X(L)P'(kX(L))$, where P' is now [15] a scaling function of j. Let us check what happens to the average multiplicity $\langle k \rangle$ if we assume that $P(k,L)$ has the above-derived form:

$$\langle k \rangle = \frac{\sum_{k=0}^{\infty} kX(L)P'(kX(L))}{\sum_{k=0}^{\infty} X(L)P'(kX(L))} = \frac{1}{CX(L)} \sum_{j=0}^{\infty} jP'(j), \quad (25)$$

where we again made the substitution $j=kX(L)$ in the sum over k. The sum in Eq. (25) is independent of L, and we finally obtain

$$\langle k \rangle \propto \frac{1}{X(L)}, \tag{26}$$

so that the whole L dependence of the average multiplicity is contained in the correction factor $X(L)$.

In this way, we have now the chance to make a cross check on our, data. If $\langle k \rangle$ is indeed size independent for $d<6$, we should be able to find a simple nonlogarithmic correction factor $X(L)=1+\text{const}/L$, such that $\langle k \rangle \propto 1/X(L)$ and consistently $P(k,L)/X(L)$ is a scaling function of the variable $j=kX(L)$. We remark that only eventual discrepancies of the rescaled histograms at large k can lead to infi-
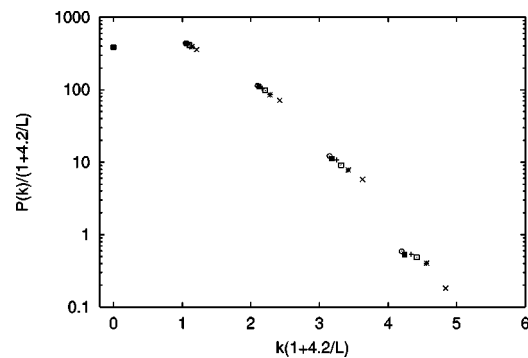

FIG. 8. Four-dimensional histogram, MBC's. Occurrence $P(k,L)$ of samples with k spanning clusters each, versus k, with the finite-size correction factor $1+4.2/L$ derived from the fit of the average multiplicity. The data sets have a variable number of samples, mostly 50 000; we normalized them all to 1000 iterations. The nice scaling is consistent with the size independence of the average multiplicity. The lattices we have taken are 20^4, 30^4, 40^4, 50^4, 70^4, and 85^4.

nitely many spanning clusters in the limit $L \to \infty$: if, for all L, there were a finite k^* beyond which the histograms collapse, $\langle k \rangle$ would necessarily approach a constant for large L, modulo finite-size corrections.

We tried then to fit the multiplicity curves in four and five dimensions with the simple two-parameter ansatz $aL/(L+b)$, to see whether we could reproduce the observed trends. As a first trial we took the four-dimensional data of Fig. 7 and fitted several portions of the increasing part of the curve, before the plateau. All our fits are quite good; besides, starting from the range [2:60], the fit is stable; i.e., we obtain the same values for the parameters a and b as for the fit on the full curve, within errors. This is quite interesting, because it allows us to predict quite precisely where saturation takes place, even if one analyzes values of L which lie well below the beginning of the plateau. The best fit curve, for which $a=0.78$ and $b=4.2$, is plotted in Fig. 7.

As a nice confirmation of this result, we show in Fig. 8 the corresponding histograms $P(k,L)$, where we use the value $b=4.2$ derived above for the scaling correction. We chose again on purpose only values of L before the plateau of the average multiplicity. The figure gives a nice data collapse, as we expected.

We repeated the analysis for the five-dimensional data, starting from the puzzling curve of Fig. 5. Here we find that the fits are very good and stable starting from the very beginning of the curve: the fit parameters a and b are basically fixed already in the range [2:20]; the best fit (dashed line in the figure) was obtained including all data points with significant statistics—i.e., for $2 \leqslant L \leqslant 91$, we obtain $a=3.12(6)$, $b=17.5(7)$. As one can see in Fig. 5, our simple ansatz (dashed line) describes very well the observed behavior of the data. We notice that on the logarithmic scale for L our simple scaling curve has an inflection point, exactly as the data. This inflection can be interpreted as a signal of a possible crossover from an initial logarithmic increase of the multiplicity to a successive convergence to a plateau; we now see that it is instead a natural feature of our scaling
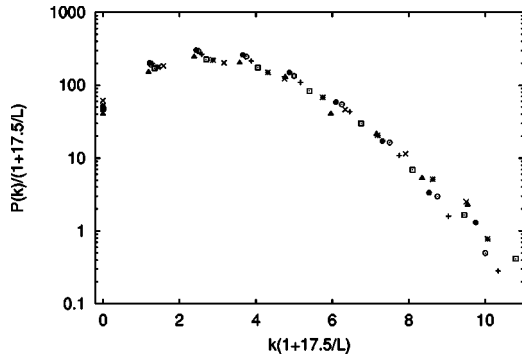
FIG. 9. Same as Fig. 8, for five-dimensional data with MBC's. One obtains again a very good scaling of the spanning cluster multiplicity distributions for $k>0$ by introducing the simple correction $1+17.5/L$. The lattices are 30^5, 40^5, 50^5, 60^5, 70^5, 80^5, and 91^5. The number of samples goes from 1000 to 50 000.

FIG. 11. Same as Fig. 9, but with PBC. The scaling is remarkable, probably due to the better statistics; the necessary correction is $1+20/L$. The lattices are 30^5, 40^5, 50^5, 60^5, 70^5, and 80^5. The number of samples goes from 8000 to 50 000.

ansatz. Notice that the correction term is more important than in four dimensions (17.5 vs 4.2). That means that the data converge much more slowly to the plateau and explains why we could not see a saturation even at $L=201$ (although the argument can be reversed). Indeed, for a given L, the ratio r of the multiplicity to the asymptotic plateau is $r=L/(L+b)$. In 4D, $r=96\%$ for $L=100$ and $r=98\%$ for $L=200$; in 5D, one would obtain 85% and 92%, respectively. In order to "see" the plateau as we do in four dimensions, we would need to go to $L\sim800$. To check the consistency of the picture in 5D we studied the scaling of the histograms $P(k,L)$, with the correction constant $b=17.5$ that we determined above. The result is illustrated in Fig. 9; the scaling is quite good, except eventually for $k=0$ (but scaling laws seldom hold for small integers) and at the very end of the tail, where the statistics is too low and there are relevant fluctuations of the data points.

Finally, we analyzed the two other data sets in five dimensions—i.e., the ones relative to FBC's and PBC's. In both cases we found that our picture works: we could find a scaling correction $X(L)=1+\text{const}/L$ such that both the average multiplicity and the histograms show a clean scaling. The histograms are shown in Figs. 10 and 11; the scaling is again very good for $k>0$, with some fluctuations at the end of the tail which are likely due to the low statistics of those points.
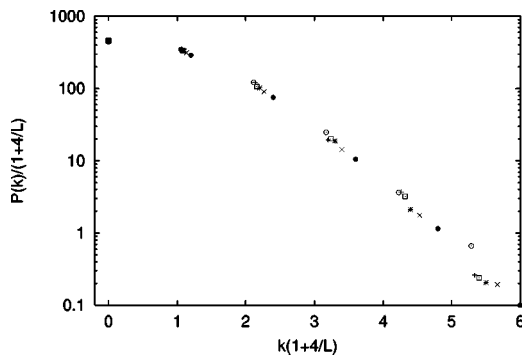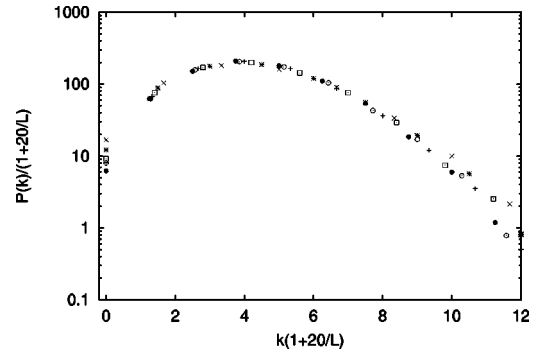

FIG. 10. Same as Fig. 9, but with FBC's. The correction is $1+4/L$, the lattices are 20^5, 30^5, 40^5, 50^5, 60^5, and 70^5. The number of samples goes from 3000 to 50 000.

We also checked for five dimensions whether other percolation quantities behave unusually and found that they do not. The size of the largest cluster at the percolation threshold varies asymptotically as $L^{d-\beta/\nu}$ and the "mean" cluster size as $L^{\gamma/\nu}$, where $\beta/\nu\simeq1.46$, $\gamma/\nu\simeq2.07$ are expected [3] in five dimensions. We found that corrections to scaling play an important role in this range of L (from 10 to 80); all our fitting curves include a correction term, for which we fixed the value of the exponent ω to the estimate 0.53 given in Ref. [16]. Taking into account this correction, the finite-size scaling fits are remarkable for all percolation variables if we use the PBC data, for which we get $\beta/\nu=1.45(2)$ and $\gamma/\nu=2.08(2)$. For MBC's the fits are also very good, but a bit worse as far as the χ^2 and the values of the exponents are concerned [$\beta/\nu=1.45(2)$, $\gamma/\nu=2.10(2)$]; for the FBC's the fits are not so good and the values of the exponents not in agreement with expectations, though quite close. Also series expansions [16], which are independent of any lattice size, gave in five dimensions the usual exponents in agreement with expectations, without indications of particular difficulties.

III. CONCLUSIONS

In summary, we have derived the scaling behavior at threshold of the average number $\langle k\rangle$ of spanning clusters with the linear dimension L of the lattice, for any space dimension d. Below the upper critical dimension $d_c=6$, $\langle k\rangle$ should approach a constant when $L\to\infty$, for $d=6$ it should increase as $\ln L$, and for $d>6$ it could increase as L^{d-6}, but could also approach a constant (depending on boundary conditions and yet unknown theoretical details). While the latter conclusions might seem just a confirmation of previous results on the topic, our work highlights two new important issues: (i) the possibility of other scaling behaviors of $\langle k\rangle$ above the upper critical dimension, which possibly depend on the boundary conditions and (ii) the relevance of corrections to scaling, which may affect the scaling behaviors up to two-three orders of magnitude in L.

Our numerical investigations confirm that the multiplicity indeed converges to a constant for $d<6$. For the case $d=5$, where we do not see a plateau even for the largest L we have

taken, a simple linear correction to scaling is able to reproduce the observed data pattern. In six dimensions the results can be made consistent with theory by adding a logarithmic finite-size correction; in more than six dimensions, both the data on the multiplicity k and those on the order parameter P_{span} and the mean cluster size S lead to very different values of the finite-size scaling exponents for different sets of boundary conditions, and we are not able to derive reliable conclusions. So if our numerical evidence below d_c is conclusive, to close the issue above d_c simulations at much larger L seem to be necessary.

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